Some Diophantine equations with small integer solutions

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In this paper we consider Diophantine equations of the type $a^2 + b^2 + c^2 = d^3 + e^3$ and a few others. Examples of solutions are given. We give parametric solutions.

I. INTRODUCTION

Examples of equations with solutions.

$$a^2 + b^2 + c^2 = d^3 + e^3 \tag{1}$$

Solutions:

$$3^2 + 6^2 + 9^2 = 1^3 + 5^3 = 126$$
$$3^2 + 1^2 + 5^2 = 2^3 + 3^3 = 35$$
$$3^2 + 1^2 + 9^2 = 3^3 + 4^3 = 91$$
$$4^2 + 9^2 + 6^2 = 5^3 + 2^3 = 133$$

$$a^2 + b^2 + c^2 = d^3 + e^3 + f^3 \tag{2}$$

Solutions:

$$1^2 + 5^2 + 6^2 = 2^3 + 3^3 + 3^3 = 62$$
$$1^2 + 7^2 + 7^2 = 2^3 + 3^3 + 4^3 = 99$$
$$1^2 + 4^2 + 8^2 = 3^3 + 3^3 + 3^3 = 81$$
$$1^2 + 6^2 + 9^2 = 3^3 + 3^3 + 4^3 = 118$$

$$a^2 + b^2 + c^2 = d^4 + e^4 + f^4 \tag{3}$$

Solutions:

$$3^2 + 5^2 + 8^2 = 1^4 + 2^4 + 3^4 = 98$$

$$a^3 + b^3 + c^3 = d^4 + e^4 + f^4 \tag{4}$$

Solutions:

$$3^3 + 5^3 + 9^3 = 4^4 + 5^4 = 881$$
$$2^3 + 5^3 + 5^3 = 1^4 + 4^4 + 1^4 = 258$$
$$2^3 + 4^3 + 6^3 = 2^4 + 2^4 + 4^4 = 288$$
$$3^3 + 7^3 + 8^3 = 1^4 + 4^4 + 5^4 = 882$$

$$a^3 + b^3 + c^3 = d^5 + e^5 + f^5 \tag{5}$$

Solutions:

$$3^3 + 4^3 + 6^3 = 2^5 + 3^5 + 2^5 = 307$$
$$3^3 + 8^3 + 9^3 = 1^5 + 3^5 + 4^5 = 1268$$

II. PARAMETRIC SOLUTIONS

The examples we provided in the previous section illustrate that the solutions exist, and probably many of them, but don't prove even that the number of solutions is infinite.

A. Some Parametric Solutions

Every particular solution may be made into a family by adding coefficients u^k to the terms.
Every solution of (1)

$$a^2 + b^2 + c^2 = d^3 + e^3$$

generates a family of solutions

$$(au^3)^2 + (bu^3)^2 + (cu^3)^2 = (du^2)^3 + (eu^2)^3, \tag{6}$$

where $u = 1, 2, 3, ...$ E.g., solution

$$3^2 + 6^2 + 9^2 = 1^3 + 5^3$$

generates a family of solutions

$$\begin{aligned}&(3u^3)^2 + (6u^3)^2 + (9u^3)^2 = (1u^2)^3 + (5u^2)^3 :\\ &24^2 + 48^2 + 72^2 = 4^3 + 20^3,\\ &81^2 + 162^2 + 243^2 = 9^3 + 45^3,\\ &...\end{aligned}$$

Every solution of (2)

$$a^2 + b^2 + c^2 = d^3 + e^3 + f^3$$

generates a family of solutions

$$(au^3)^2 + (bu^3)^2 + (cu^3)^2 = (du^2)^3 + (eu^2)^3 + (fu^2)^3. \tag{7}$$

Every solution of (3)

$$a^2 + b^2 + c^2 = d^4 + e^4 + f^4$$

generates a family of solutions

$$(au^2)^2 + (bu^2)^2 + (cu^2)^2 = (du)^3 + (eu)^3 + (fu)^3. \tag{8}$$

Every solution of (4)

$$a^3 + b^3 + c^3 = d^4 + e^4 + f^4$$

generates a family of solutions

$$(au^4)^3 + (bu^4)^3 + (cu^4)^3 = (du^3)^4 + (eu^3)^4 + (fu^3)^4, \tag{9}$$

Every solution of (5)

$$a^3 + b^3 + c^3 = d^5 + e^5 + f^5$$

generates a family of solutions

$$(au^5)^3 + (bu^5)^3 + (cu^5)^3 = (du^3)^5 + (eu^3)^5 + (fu^3)^5, \tag{10}$$

B. Splitting into Parts and Parametrization

The equations may be split into parts, and each part parametrized.
The equation (1)

$$a^2 + b^2 + c^2 = d^3 + e^3$$

may be split as

$$a^2 + b^2 = d^3 \tag{11}$$
$$c^2 = e^3 \tag{12}$$

and parametrized:

$$a = m(m^2 + n^2) \tag{13}$$
$$b = n(m^2 + n^2) \tag{14}$$
$$d = (m^2 + n^2) \tag{15}$$
$$c = u^3 \tag{16}$$
$$e = u^2 \tag{17}$$

The equation (2)

$$a^2 + b^2 + c^2 = d^3 + e^3 + f^3$$

may be split as

$$a^2 + b^2 = d^3 \tag{18}$$
$$c^2 = e^3 + f^3 \tag{19}$$

and parametrized:

$$a = m(m^2 + n^2) \tag{20}$$
$$b = n(m^2 + n^2) \tag{21}$$
$$d = (m^2 + n^2) \tag{22}$$
$$e = p(p^3 + q^3) \tag{23}$$
$$f = q(p^3 + q^3) \tag{24}$$
$$c = (p^3 + q^3)^2 \tag{25}$$

For equation (3)

$$a^2 + b^2 + c^2 = d^4 + e^4 + f^4$$

such approach won't work here because diophantine equation $x^4 + y^4 = z^2$ has no all nonzero solutions.
The equation (4)

$$a^3 + b^3 + c^3 = d^4 + e^4 + f^4$$

may be split as

$$a^3 + b^3 = d^4 \tag{26}$$
$$c^3 = e^4 + f^4 \tag{27}$$

and parametrized:

$$a = m(m^3 + n^3) \tag{28}$$
$$b = n(m^3 + n^3) \tag{29}$$
$$d = (m^3 + n^3) \tag{30}$$
$$e = p(p^4 + q^4)^2 \tag{31}$$
$$f = q(p^4 + q^4)^2 \tag{32}$$
$$c = (p^4 + q^4)^3 \tag{33}$$

The equation (5)

$$a^3 + b^3 + c^3 = d^5 + e^5 + f^5$$

may be split as

$$a^3 + b^3 = d^5 \quad (34)$$
$$c^3 = e^5 + f^5 \quad (35)$$

and parametrized:

$$a = m(m^3 + n^3)^3 \quad (36)$$
$$b = n(m^3 + n^3)^3 \quad (37)$$
$$d = (m^3 + n^3)^2 \quad (38)$$
$$e = p(p^5 + q^5) \quad (39)$$
$$f = q(p^5 + q^5) \quad (40)$$
$$c = (p^5 + q^5)^2 \quad (41)$$

III. CONCLUSIONS

In this work we studied Diophantine equations of the type $a^3 + b^3 + c^3 = d^5 + 1$, as well as related equations $a^3 + b^3 + c^3 = d^5$ and $a^3 + b^3 = c^3$. Examples of solutions, as well as parametric solutions were given.
Thanks who helped...
References:

[1] ???