SOME ALMOST PARTITION THEORETIC IDENTITIES

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Abstract. We give some new q series identities that resemble the traditional q series partition generating functions.

1. Introduction

The literature on q series goes way back to the nineteenth century, starting with Heine [4] and [5], in 1847 and 1878. This generalized the classical hypergeometric series work introduced by in 1813 by Gauss [3]. We give some new identities that are similar to traditional q series.

Euler transform of an integer sequence $\{a_k\}$ is given by [6]

$$\sum_{k=0}^{\infty} b_k q^k = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{a_k}}, \quad b_0 = 1. \tag{1.1}$$

We are considering constant sequence $\{a_k\} = \{1/n\}$, for which Euler transform is

$$\sum_{k=0}^{\infty} b_k q^k = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{1/n}}. \tag{1.2}$$

We give identities for q series

$$\sum_{k=0}^{\infty} g_k q^k = \prod_{k=1}^{\infty} \frac{1}{(1-(nq)^k)^{1/n}} \tag{1.3}$$

and connection of them to Euler transform (1.2).

Our first identity is related to Euler transform of $\{1/2\}$

$$\sum_{k=1}^{\infty} b_k q^k = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{1/2}} = 1+\frac{1}{2}q+\frac{7}{8}q^2+\frac{17}{16}q^3+\frac{203}{128}q^4+\frac{455}{256}q^5+\frac{2723}{1024}q^6+.... \tag{1.4}$$

The identity is given by

Theorem 1.1.

$$\prod_{k=1}^{\infty} \sqrt{\frac{1}{1-(2q)^k}} = \sum_{k=0}^{\infty} g_k q^k = 1+q+\frac{7}{2}q^2+\frac{17}{2}q^3+\frac{203}{8}q^4+\frac{455}{8}q^5+\frac{2723}{16}q^6+..., \tag{1.5}$$

where the numerators of g_k $1, 1, 7, 17, 203, 455, 2723, ...$ are identical to numerators of b_k of Euler transform of $\{1/2\}$ (1.4), and the denominators of g_k are the greatest power of 2 in $k!$.

First we'll prove

Lemma 1.1. *The greatest power of prime p in $k(k+1)...(pk)$ is k.*

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PROOF. The greatest power of prime p in $k(k+1)...(pk)$ is

(1.6)
$$\sum_{i=1}^{\infty}\left(\left[\frac{pk}{p^i}\right]-\left[\frac{k}{p^i}\right]\right) = \left[\frac{pk}{p}\right]-\left[\frac{k}{p}\right]+\left[\frac{k}{p}\right]-\left[\frac{k}{p^2}\right]+\left[\frac{k}{p^2}\right]-\left[\frac{k}{p^3}\right]+... = \left[\frac{pk}{p}\right]=k.$$

END OF PROOF.

PROOF of the theorem (1.1). The q series (1.5) differ from the Euler transform (1.4) by having $(2q)$ instead of q, so $g_kq^k = b_k(2q)^k$, and hence $g_k = 2^kb_k$. Denominators of b_k are 2^{d_k}, where 2^{d_k} = power of 2 in $(2k)!$ [7]. According to the lemma (1.1, 2^k = the greatest power of 2 in $(k+1)(k+2)...(2k)$. Therefore, multiplying b_k by 2^k to obtain g_k reduces denominator of b_k from being the greatest power of 2 in $(2k)!$ to the greatest power of 2 in $k!$ Since $2^k \leq$ denominator of b_k, the numerator of g_k stays the same as in b_k. END OF PROOF.

Similarly, our second identity is related to Euler transform of $\{1/3\}$

$$(1.7)\quad \sum_{k=1}^{\infty} b_kq^k = \prod_{k=1}^{\infty}\frac{1}{(1-q^k)^{1/3}} = 1+\frac{1}{3}q+\frac{5}{9}q^2+\frac{50}{81}q^3+\frac{215}{243}q^4+\frac{646}{729}q^5+\frac{8711}{6561}q^6+....$$

The identity is given by

Theorem 1.2.

$$(1.8)\quad \prod_{k=1}^{\infty}\sqrt[3]{\frac{1}{1-(3q)^k}} = \sum_{k=0}^{\infty} g_kq^k = 1+q+5q^2+\frac{50}{3}q^3+\frac{215}{3}q^4+\frac{646}{3}q^5+\frac{8711}{9}q^6+...,$$

where the numerators of g_k $1, 1, 5, 50, 215, 646, 8711, ...$ *are identical to numerators of* b_k *of Euler transform of* $\{1/3\}$ *(1.7), and the denominators of* g_k *are the greatest power of* 3 *in* $k!$.

PROOF. The q series (1.8) differ from the Euler transform (1.7) by having $(3q)$ instead of q, so $g_kq^k = b_k(3q)^k$, and hence $g_k = 3^kb_k$. Denominators of b_k are 3^{d_k}, where 3^{d_k} = power of 3 in $(3k)!$ [8]. According to the lemma (1.1, 3^k = the greatest power of 3 in $(k+1)(k+2)...(3k)$. Therefore, multiplying b_k by 3^k to obtain g_k reduces denominator of b_k from being the greatest power of 3 in $(3k)!$ to the greatest power of 3 in $k!$ Since $3^k \leq$ denominator of b_k, the numerator of g_k stays the same as in b_k. END OF PROOF.

The OEIS references (see Sloane [7] and [8]) indicate that the above two identities are not known in the form presented here.

The results (1.5) and (1.8) may be generalized in

Conjecture 1.3. *If canonical representation of positive integer* n *is* $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_m^{\alpha_m}$, *then*

$$(1.9)\qquad \prod_{k=1}^{\infty}\sqrt[n]{\frac{1}{1-(nq)^k}} = \sum_{k=0}^{\infty} g_kq^k = 1+q+\frac{a_2}{c_2}q^2+\frac{a_3}{c_3}q^3+\frac{a_4}{c_4}q^4+...,$$

where the numerators of g_k $1, 1, a_2, a_3, a_4, ...$ *are identical to numerators of* b_k *of Euler transform of* $\{1/n\}$ *(1.2), and the denominators of* g_k $1, 1, c_2, c_3, c_4, ...$ *are the products of the greatest powers of* p_i *in* $k!$.

PARTIAL PROOF. We'll prove the conjecture for $n = p_1p_2...p_m$, or n being square-free. The q series (1.9) differ from the Euler transform (1.2) by having (nq) instead of q, so $g_kq^k = b_k(nq)^k$, and hence $g_k = n^kb_k = p_1^kp_2^k...p_m^kb_k$. Denominators of b_k are $(p_1^{d_{1k}}p_2^{d_{2k}}...p_m^{d_{mk}})$, where $p_i^{d_{ik}}$ = the greatest power of p_i in $(p_ik)!$. According to the lemma (1.1), for prime p, p_i^k = power of p_i in $(k+1)(k+2)...(p_ik)$. Therefore, multiplying b_k by $(p_1^k p_2^k ... p_m^k)$ to obtain g_k reduces denominator of b_k from being the product of the greatest powers of p_i in $(p_ik)!$ to the product of the greatest powers of p_i in $k!$. END OF PARTIAL PROOF.

An example with non-prime n:

(1.10)
$$\prod_{k=1}^{\infty} \sqrt[6]{\frac{1}{1-(6q)^k}} = 1+q+\frac{19}{2}q^2+\frac{343}{6}q^3+\frac{11305}{24}q^4+\frac{58349}{24}q^5+\frac{3230255}{144}q^6+\frac{15652637}{144}q^7+...,$$

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