SOME LEFT NESTED RADICALS

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Abstract. Most of the literature deals with infinitely right nested radicals. We consider here examples of left nested radicals.

1. Introduction

The general form of a right nested nested radical is

\[
\sqrt{a_1 + c_1 + \sqrt{a_2 + c_2 + \sqrt{a_3 + c_3 + \ldots}}} \tag{1.1}
\]

Conversely, the general form of a left nested nested radical is (\cite{1},\cite{2},\cite{3})

\[
\sqrt{\ldots + c_5 + \sqrt{a_4 + c_4 + \sqrt{a_3 + c_3 + \ldots}}} \tag{1.2}
\]

It is easy to see that if the sequences \(\{r_i, a_i, c_i\}\) are periodic, and the sequences in (1.1) and (1.2) are reverses of each other, then they are both converge to the same number. (An additional condition for period larger than 1 - partial radicals for (1.2) need to be taken at the end of the period.)

We here are considering non-periodic nested radicals. We’ll work with two examples of left nested radicals as counterparts of the two known right nested radicals.

2. Example 1

The nested radical

\[
\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \ldots}}}} \tag{2.1}
\]

is well known and it converges to 1.757932756618..., known as "nested radical constant" (\cite{4}, \cite{5}).

The analogous left nested radical

\[
\sqrt{\ldots + n + \sqrt{4 + \sqrt{3 + \sqrt{2 + 1} + \ldots}}} \tag{2.2}
\]

is evidently diverging. Still, we can study partial radicals

\[
s_n = \sqrt{n + \ldots + \sqrt{4 + \sqrt{3 + \sqrt{2 + 1} + \ldots}}} \tag{2.3}
\]

1991 Mathematics Subject Classification. Primary: 11D25; Secondary: 11D45, 11D41.

Key words and phrases. Cubic and quartic equations, Counting solutions of Diophantine equations, Higher degree equations, Fermats equation.
Looking at the dominant term of the expression, we conclude, that
\begin{equation}
    s_n \sim \sqrt{n}
\end{equation}

or
\begin{equation}
    s_n = \sqrt{n}(1 + o(1)).
\end{equation}

Further, we can prove

**Theorem 2.1.**
\begin{equation}
    \lim_{n \to \infty} (s_n - \sqrt{n}) = \frac{1}{2}.
\end{equation}

**PROOF.**
\begin{equation}
    (s_n - \sqrt{n}) = \frac{s_n^2 - n}{s_n + \sqrt{n}} = \frac{s_{n-1} - n}{s_n + \sqrt{n}} = \frac{1}{2} + o(1).
\end{equation}

END OF PROOF.

We can generalize this result from nested square roots to an arbitrary power \( p \).

**Theorem 2.2.** Let
\begin{equation}
    s_n = (n + \ldots + (4 + (3 + (2 + 1)^p)^p)^p \ldots)^p.
\end{equation}

Then
\begin{equation}
    \lim_{n \to \infty} (s_n - n^p) = \begin{cases} 
    0, & p < \frac{1}{2} \\
    \frac{1}{2}, & p = \frac{1}{2} \\
    +\infty, & p > \frac{1}{2}
\end{cases}
\end{equation}

**PROOF.**
\begin{equation}
    (s_n - n^p) \sim (n + (n - 1)^p - n^p \sim \left( n \left( 1 + \frac{1}{n^{1-p}} \right) \right)^p - n^p \sim
\end{equation}
\begin{equation}
    n^p \left( 1 + \frac{p}{n^{1-p}} \right) - n^p = pn^{2p-1} \to \begin{cases} 
    0, & p < \frac{1}{2} \\
    \frac{1}{2}, & p = \frac{1}{2} \\
    +\infty, & p > \frac{1}{2}
\end{cases}
\end{equation}

END OF PROOF.

3. Example 2 - a variation on Ramanujan’s nested radical

Ramanujan introduced the nested radical
\begin{equation}
    \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}}
\end{equation}

in [6]. It is well known, and it converges to 3.

The corresponding left nested radical
\begin{equation}
    \ldots + \sqrt{1 + n} \ldots + 5 \sqrt{1 + 4} \ldots 3 \sqrt{1 + 2} \sqrt{1 + \sqrt{1}} \ldots
\end{equation}
is evidently diverging. Again, we can study partial radicals

\[(3.3) \quad s_n = \sqrt{1 + n} \sqrt{1 + 4} \sqrt{1 + 3} \sqrt{1 + 2} \sqrt{1 + \sqrt{1}} ... \]

Looking at the dominant terms of the expression, we conclude, that

\[(3.4) \quad s_n \sim n^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...} = n \]

or

\[(3.5) \quad s_n = n(1 + o(1)). \]

Similarly to the Example 1, we have

**Theorem 3.1.** \(\lim_{n \to \infty} (s_n - n) = -1.\)

and as generalization

**Theorem 3.2.** Let

\[(3.7) \quad s_n = (1 + n + \ldots + 5 (1 + 4 (1 + 3 (1 + 2 (1 + 1 \ldots + p)) \ldots)))^p. \]

Then

\[(3.8) \quad \lim_{n \to \infty} (s_n - n^{1/p}) = \begin{cases} 
0, & p < \frac{1}{2} \\
-1, & p = \frac{1}{2} \\
-\infty, & p > \frac{1}{2}
\end{cases} \]

The proofs of the Theorems (3.1) and (3.2) are similar to those of the respective theorems of the Example 1.

**References**


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