On integer solutions to $w^5 + x^5 = y^5 + z^n$ for $n = 3, 4, 5, 6$

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In this paper we consider Diophantine equations of the type $w^5 + x^5 = y^5 + z^n$, with $n = 3, 4, 5, 6$. We derive an infinite set of solutions in Gaussian integers for the case $n = 5$.

I. INTRODUCTION

A famous open question [1] is the solution in positive integers of

$$w^5 + x^5 = y^5 + z^5. \quad (1)$$

While not settling this totally in integers, we give a solution with integers $w$ and $x$ where the right side $y$ and $z$ are Gaussian integers. The solutions to

$$w^n + x^n = y^n + z^n, \quad (2)$$

for $n = 4$ are well known to date back to Euler, [2] and the case where $n = 3$ is solved by the well known and celebrated "Taxicab numbers" named after the famous Hardy and Ramanujan anecdote. [3]. In this note we state some new integer cases for

$$w^5 + x^5 = y^5 + z^n, \quad (3)$$

for $n = 3, 4$ and 6.

Examples.

$$121^5 + 143^5 = 110^5 + 4114^3;$$
$$500^5 + 225^5 = 100^5 + 2375^4;$$
$$636^5 + 212^5 = 424^5 + 212^6.$$

II. GAUSSIAN INTEGER SOLUTIONS TO $a^5 + b^5 = c^5 + d^5$

We note here cases of (1) for integers $w, x$ and Gaussian integers $y, z$.

$$3^5 + 1^5 = (2 + i3)^5 + (2 - i3)^5,$$
$$13^5 + 11^5 = (12 + i17)^5 + (12 - i17)^5,$$
$$71^5 + 69^5 = (70 + i99)^5 + (70 - i99)^5,$$
$$409^5 + 407^5 = (408 + i577)^5 + (408 - i577)^5,$$
$$2379^5 + 2377^5 = (2378 + i3363)^5 + (2378 - i3363)^5.$$

(4)

Anyone familiar with the classical Pell equation $1 + 2m^2 = n^2$, will recognize the well known continued fraction convergents featured in the right hand sides of these equations, leading to a fairly easy assertion that
**Theorem 1** An infinite number of solutions to (1) are given by

\[(m + 1)^5 + (m - 1)^5 = (m + in)^5 + (m - in)^5,\]  

where

\[m = \frac{(3 + 2\sqrt{2})^a - (3 - 2\sqrt{2})^a}{2\sqrt{2}}\]  

and

\[n = \frac{(3 + 2\sqrt{2})^a + (3 - 2\sqrt{2})^a}{2}\]

for positive integers \(a\).

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### A. Parametric solutions

Every solution of

\[w^5 + x^5 = y^5 + z^3\]

(7)

generates a family of solutions

\[(wt^3)^5 + (xt^3)^5 = (yt^3)^5 + (zt^3)^3,\]

(8)

where \(t = 1, 2, 3, \ldots\)

For instance,

\[121^5 + 143^5 = 110^5 + 4114^3;\]

generates

\[968^5 + 1144^5 = 8810^5 + 131648^3,\]

\[3267^5 + 3861^5 = 2970^5 + 999702^3,\]

\[\ldots\]

Similarly, every solution of

\[w^5 + x^5 = y^5 + z^4\]

(9)

generates a family of solutions

\[(wt^4)^5 + (xt^4)^5 = (yt^4)^5 + (zt^4)^4,\]

(10)

and solution of

\[w^5 + x^5 = y^5 + z^6\]

(11)

generates a family of solutions

\[(wt^6)^5 + (xt^6)^5 = (yt^6)^5 + (zt^5)^6.\]

(12)

### B. Another set of parametric solutions

\[w^5 + x^5 = y^5 + z^3\]

A parametric solution:
\[ w = m(m^5 + n^5 - p^5) \]
\[ x = n(m^5 + n^5 - p^5) \]
\[ y = p(m^5 + n^5 - p^5) \]
\[ z = (m^5 + n^5 - p^5)^2 \]

It’s simultaneously a solution for
\[ w^5 + x^5 = y^5 + z^6 : \]

\[ w, x, y \] the same, \[ z = (m^5 + n^5 - p^5). \]

A parametric solution for
\[ w^5 + x^5 = y^5 + z^4 : \]
\[ w = (m(m^{15} + n^{15} - p^{15}))^3 \]
\[ x = (n(m^{15} + n^{15} - p^{15}))^3 \]
\[ y = (p(m^{15} + n^{15} - p^{15}))^3 \]
\[ z = (m^{15} + n^{15} - p^{15})^4 \]

Our parametric solutions show the existence of infinite number of solutions to the equations (7, 9, 11).
Note that the parametric solutions (8, 10, 12) and (13 - 14) give only a small fraction of all solutions of (7, 9, 11).

III. CONCLUSIONS

In this work we studied Diophantine equations of the type \( w^5 + x^5 = y^5 + z^n \), with \( n = 3, 4, 6 \). Examples of solutions, as well as parametric solutions were given.

Thanks who helped...

References:

[1] Find reference