GAUSSIAN INTEGER SOLUTIONS FOR THE FIFTH POWER TAXICAB NUMBER PROBLEM

GEOFFREY B CAMPBELL AND ALEKSANDER ZUJEV

Abstract. The famous open problem of finding positive integer solutions to $a^5 + b^5 = c^5 + d^5$ is considered, and related solutions are found in two distinct settings: firstly, where $a$ and $b$ are both positive integers with $c$ and $d$ both Gaussian integers; secondly, where all of $a$, $b$, $c$, and $d$ are Gaussian integers.

1. Introduction

A famous open question [3] is the solution in positive integers of

$$w^5 + x^5 = y^5 + z^5. \tag{1.1}$$

It however has solutions in Gaussian integers. A few examples:

$$\begin{align*}
(1 + 2i)^5 + (4 - 7i)^5 &= (-3 - 6i)^5 + (8 + i)^5 \\
(3 + 10i)^5 + (9 - 6i)^5 &= (6 + 5i)^5 + (6 - i)^5 \\
(4 + 45i)^5 + (36 - 5i)^5 &= (45 + 4i)^5 + (-5 + 36i)^5
\end{align*} \tag{1.2-1.4}$$

While not resolving this for integers generally, we give an infinite set of solutions with integers $w$ and $x$ where the right side $y$ and $z$ are Gaussian integers. Also, we give an infinite set of solutions where all of $w$, $x$, $y$, and $z$ are Gaussian integers. The solutions to

$$w^n + x^n = y^n + z^n, \tag{1.5}$$

for $n = 4$ are well known and date back to Euler [5], and the case where $n = 3$ is solved by the well known and celebrated ”Taxicab numbers” named after the famous Hardy and Ramanujan anecdote. (See Hardy [2].)

2. Solutions where $a$ and $b$ are both positive integers with $c$ and $d$ both Gaussian integers

Our first result is encapsulated in the

**Theorem 2.1.** If the Pell number sequence is 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378,...; then an infinite sequence of solutions to (1.1) is given by:

$$\begin{align*}
3^5 + 1^5 &= (2 + i3)^5 + (2 - i3)^5, \\
13^5 + 11^5 &= (12 + i17)^5 + (12 - i17)^5, \\
71^5 + 69^5 &= (70 + i99)^5 + (70 - i99)^5, \\
409^5 + 407^5 &= (408 + i577)^5 + (408 - i577)^5
\end{align*} \tag{2.1-2.4}$$

1991 Mathematics Subject Classification. Primary: 11D41; Secondary: 11D45, 11Y50.

Key words and phrases. Diophantine Higher degree equations; Fermat’s equation, Counting solutions of Diophantine equations, Counting solutions of Diophantine equations.
Theorem 3.3. The equation

\[ (3.2) \]

\[ (a+bi)^5 + (c+di)^5 = (b+ai)^5 + (d+ci)^5 \]

has an infinite number of solutions.

PROOF. The equation (3.2) is equivalent to

\[ (3.3) \]

\[ (a-b)^5 - 20a^2b^2(a-b) + (c-d)^5 - 20c^2d^2(c-d) = 0, \]

which is satisfied if \( a-b+c-d = 0 \) and \( ab+cd = 0 \). Let \( a \) be arbitrary. Substituting \( b = a+c-d \) from the first equation into the second, we get \( d = a+2ac/(a-c) \), which is integer at least for \( c = 0, a \pm 1, a \pm 2, 2a, a \pm 2a \). So for every integer \( a \) equation (3.2) has a few solutions, altogether amounting to an infinite number. END OF PROOF.

A very similar

Theorem 3.3. The equation

\[ (3.4) \]

\[ (a+bi)^5 + (c+di)^5 = (b+ai)^5 + (d+ci)^5 \]

has an infinite number of solutions.

Examples of solutions in the shape of (3.2) and (3.4):

\[ (3.5) \]

\[ (3+28i)^5 + (4-21i)^5 = (28+3i)^5 + (-21+4i)^5 \]

\[ (3.6) \]

\[ (2+3i)^5 + (3+2i)^5 = (6-i)^5 + (-1+6i)^5 \]
REFERENCES


MATHEMATICAL SCIENCES INSTITUTE, THE AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT, 0200, AUSTRALIA
E-mail address: Geoffrey.Campbell@anu.edu.au

DEPARTMENT OF PHYSICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA, USA
E-mail address: azujev@ucdavis.edu