SOME LEFT NESTED RADICALS

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Abstract. Most of the literature deals with infinitely **right** nested radicals. We consider here examples of **left** nested radicals.

1. Introduction

The general form of a right nested nested radical is

$$\sqrt[r_1]{a_1 + c_1 \sqrt[r_2]{a_2 + c_2 \sqrt[r_3]{a_3 + c_3 \sqrt[r_4]{a_4 + ...}}}} \tag{1.1}$$

Conversely, the general form of a left nested nested radical is ([1],[2],[3])

$$... + c_5 \sqrt[r_4]{a_4 + c_4 \sqrt[r_3]{a_3 + c_3 \sqrt[r_2]{a_2 + c_2 \sqrt[r_1]{a_1}}}} \tag{1.2}$$

It is easy to see that if the sequences $\{r_i, a_i, c_i\}$ are periodic, and the sequences in (1.1) and (1.2) are reverses of each other, then they are both converge to the same number. (An additional condition for period larger than 1 - partial radicals for (1.2) need to be taken at the end of the period.)

We here are considering non-periodic nested radicals. We'll work with two examples of left nested radicals as counterparts of the two known right nested radicals.

2. Example 1

The nested radical

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + ...}}}} \tag{2.1}$$

is well known and it converges to 1.757932756618..., known as "nested radical constant" ([4], [5]).

The analogous left nested radical

$$... + \sqrt{n + ... + \sqrt{4 + \sqrt{3 + \sqrt{2 + 1\sqrt{1}}}}...} \tag{2.2}$$

is evidently diverging. Still, we can study partial radicals

$$s_n = \sqrt{n + ... + \sqrt{4 + \sqrt{3 + \sqrt{2 + 1\sqrt{1}}}}...} \tag{2.3}$$

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Looking at the dominant term of the expression, we conclude, that

$$s_n \sim \sqrt{n} \tag{2.4}$$

or

$$s_n = \sqrt{n}(1 + o(1)). \tag{2.5}$$

Further, we can prove

Theorem 2.1.

$$\lim_{n->\infty} (s_n - \sqrt{n}) = \frac{1}{2} \tag{2.6}$$

PROOF.

$$(s_n - \sqrt{n}) = \frac{s_n^2 - n}{s_n + \sqrt{n}} = \frac{s_{n-1}}{s_n + \sqrt{n}} = \frac{1}{2} + o(1). \tag{2.7}$$

END OF PROOF.

We can generalize this result from nested square roots to an arbitrary power p.

Theorem 2.2. *Let*

$$s_n = (n + ... + (4 + (3 + (2 + 1(1)^p)^p)^p)^p ...)^p. \tag{2.8}$$

Then

$$\lim_{n\to\infty} (s_n - n^p) = \begin{cases} 0, & p < \frac{1}{2} \\ \frac{1}{2}, & p = \frac{1}{2} \\ +\infty, & p > \frac{1}{2} \end{cases} \tag{2.9}$$

PROOF.

$$(s_n - n^p) \sim (n + (n-1)^p)^p - n^p \sim \left(n \left(1 + \frac{1}{n^{1-p}} \right) \right)^p - n^p \sim \tag{2.10}$$

$$n^p \left(1 + \frac{p}{n^{1-p}} \right) - n^p = pn^{2p-1} \rightarrow \begin{cases} 0, & p < \frac{1}{2} \\ \frac{1}{2}, & p = \frac{1}{2} \\ +\infty, & p > \frac{1}{2} \end{cases}$$

END OF PROOF.

3. Example 2 - a variation on Ramanujan's nested radical

Ramanujan introduced the nested radical

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + ...}}}} \tag{3.1}$$

in [6]. It is well known, and it converges to 3.

The corresponding left nested radical

$$... + \sqrt{1 + n\sqrt{... + 5\sqrt{1 + 4\sqrt{1 + 3\sqrt{1 + 2\sqrt{1 + \sqrt{1}...}}}}}} \tag{3.2}$$

is evidently diverging. Again, we can study partial radicals

$$s_n = \sqrt{1 + n\sqrt{... + 5\sqrt{1 + 4\sqrt{1 + 3\sqrt{1 + 2\sqrt{1 + \sqrt{1}...}}}}}} \tag{3.3}$$

Looking at the dominant terms of the expression, we conclude, that

$$s_n \sim n^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...} = n \tag{3.4}$$

or

$$s_n = n(1 + o(1)). \tag{3.5}$$

Similarly to the Example 1, we have

Theorem 3.1.

$$\lim_{n->\infty} (s_n - n) = -1. \tag{3.6}$$

and as generalization

Theorem 3.2. *Let*

$$s_n = \left(1 + n\left(+... + 5\left(1 + 4\left(1 + 3\left(1 + 2\left(1 + 1^p\right)^p\right)^p\right)^p\right)^p ...\right)^p\right)^p. \tag{3.7}$$

Then

$$\lim_{n\to\infty} \left(s_n - n^{\frac{p}{1-p}}\right) = \begin{cases} 0, & p < \frac{1}{2} \\ -1, & p = \frac{1}{2} \\ -\infty, & p > \frac{1}{2} \end{cases} \tag{3.8}$$

The proofs of the Theorems (3.1) and (3.2) are similar to those of the respective theorems of the Example 1.

References

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