Product of digits of a cube, itself a cube

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In this paper we consider cubes of integers in base 10, such that the product of digits of this cube is itself a cube. We consider asymptotic density of solutions. We also compare the results with ones with bases different from 10.
Need editing / adding

I. INTRODUCTION

Consider the cube of an integer in base 10. How often is the product of digits of this cube, itself a cube? (Exclude cases with zero among the digits of the cube of the starting number.) For example, the cube of 29 is 24389. The product of digits 2·4·3·8·9 is 1728, which is 12 cubed. This question seems to have a non-trivial answer that is not in the literature. We think there are probably theorems close to the surface which resolve this partially.
Technically, the problem may be stated in terms of Diophantine equations:
Solve in integers

$$10^n a_n + 10^{n-1} a_{n-1} + ... + a_0 \ = \ x^3$$
$$a_n \cdot a_{n-1} \cdot ... \cdot a_0 \ = \ y^3$$

where $0 \ < \ a_n, \ a_{n-1}, \ ... \ , \ a_0 \ < \ 10$.

II. MAIN RESULTS

A. Computational results

Numbers under 1000 such that the product of digits of their cube is itself a cube:

$n = 1 \ \ 1^3 = 1 \ \ product = 1 = 1^3$
$n = 2 \ \ 2^3 = 8 \ \ product = 8 = 2^3$
$n = 3 \ \ 29^3 = 24389 \ \ product = 1728 = 12^3$
$n = 4 \ \ 61^3 = 226981 \ \ product = 1728 = 12^3$
$n = 5 \ \ 211^3 = 9393931 \ \ product = 19683 = 27^3$
$n = 6 \ \ 224^3 = 11239424 \ \ product = 1728 = 12^3$
$n = 7 \ \ 259^3 = 17373979 \ \ product = 250047 = 63^3$
$n = 8 \ \ 331^3 = 36264691 \ \ product = 46656 = 36^3$
$n = 9 \ \ 406^3 = 66923416 \ \ product = 46656 = 36^3$
$n = 1 \ \ 0456^3 = 94818816 \ \ product = 110592 = 48^3$
$n = 1 \ \ 1704^3 = 348913664 \ \ product = 373248 = 72^3$
$n = 1 \ \ 2758^3 = 435519512 \ \ product = 27000 = 30^3$
$n = 1 \ \ 3774^3 = 463684824 \ \ product = 884736 = 96^3$
$n = 1 \ \ 4819^3 = 549353259 \ \ product = 729000 = 90^3$
$n = 1 \ \ 5822^3 = 555412248 \ \ product = 64000 = 40^3$
$n = 1 \ \ 6906^3 = 743677416 \ \ product = 592704 = 84^3$

B. Bases other than 10

We wondered, what happens in other than 10 bases?

Base = 2
Here there are no solutions except trivial 1, Because we need $k^3 = (11...1)_{b2} = 2^m - 1$, which isn't possible according to Catalan's conjecture (Mihailescu's theorem).
Next are a few first results in the bases 3 to 9, cubes are given in decimal and base.

Base = 3

$$n=1 \;\; 1^3 = 1 \;\; (1)_{b3} \;\; product = 1 = 1^3$$
$$n=2 \;\; 32^3 = 32768 \;\; (1122221122)_{b3} \;\; product = 64 = 4^3$$

Need to prove it, or give some good reasoning
Surprize! The only solution we found is $32^3 = (2^5)^3 = 2^{15}$. Doesn't look as an accident. Need to prove it. We checked n up to $7 \cdot 10^6$, and there were no more solutions; Also checked all $n = 2^k$, k up to 2048, and no solutions, except already known $k = 5$.
It is understood why the product of digits, which are 1 or 2, is a power of 2.

Base = 4

$$n=1 \;\; 1^3 = 1 \;\; (1)_{b4} \;\; product = 1 = 1^3$$
$$n=2 \;\; 35^3 = 42875 \;\; (22131323)_{b4} \;\; product = 216 = 6^3$$
$$n=3 \;\; 73^3 = 389017 \;\; (1132332121)_{b4} \;\; product = 216 = 6^3$$
$$n=4 \;\; 85^3 = 614125 \;\; (2111323231)_{b4} \;\; product = 216 = 6^3$$
$$n=5 \;\; 777^3 = 469097433 \;\; (123331131233121)_{b4} \;\; product = 5832 = 18^3$$

Base = 5

$$n=1 \;\; 1^3 = 1 \;\; (1)_{b5} \;\; product = 1 = 1^3$$
$$n=2 \;\; 33^3 = 35937 \;\; (2122222)_{b5} \;\; product = 64 = 4^3$$
$$n=3 \;\; 166^3 = 4574296 \;\; (2132334141)_{b5} \;\; product = 1728 = 12^3$$
$$n=4 \;\; 204^3 = 8489664 \;\; (4133132124)_{b5} \;\; product = 1728 = 12^3$$
$$n=5 \;\; 263^3 = 18191447 \;\; (14124111242)_{b5} \;\; product = 512 = 8^3$$

Base = 6

$$n=1 \;\; 1^3 = 1 \;\; (1)_{b6} \;\; product = 1 = 1^3$$
$$n=2 \;\; 8^3 = 512 \;\; (2212)_{b6} \;\; product = 8 = 2^3$$
$$n=3 \;\; 15^3 = 3375 \;\; (23343)_{b6} \;\; product = 216 = 6^3$$
$$n=4 \;\; 40^3 = 64000 \;\; (1212144)_{b6} \;\; product = 64 = 4^3$$
$$n=5 \;\; 237^3 = 13312053 \;\; (1153153513)_{b6} \;\; product = 3375 = 15^3$$

Base = 7

$$n=1 \;\; 1^3 = 1 \;\; (1)_{b7} \;\; product = 1 = 1^3$$
$$n=2 \;\; 2^3 = 8 \;\; (11)_{b7} \;\; product = 1 = 1^3$$
$$n=3 \;\; 90^3 = 729000 \;\; (6124236)_{b7} \;\; product = 1728 = 12^3$$
$$n=4 \;\; 108^3 = 1259712 \;\; (13464426)_{b7} \;\; product = 13824 = 24^3$$
$$n=5 \;\; 200^3 = 8000000 \;\; (124666421)_{b7} \;\; product = 13824 = 24^3$$

Base = 8

$$n=1 \;\; 1^3 = 1 \;\; (1)_{b8} \;\; product = 1 = 1^3$$
$$n=2 \;\; 43^3 = 79507 \;\; (233223)_{b8} \;\; product = 216 = 6^3$$
$$n=3 \;\; 463^3 = 99252847 \;\; (572475157)_{b8} \;\; product = 343000 = 70^3$$
$$n=4 \;\; 755^3 = 430368875 \;\; (3151564153)_{b8} \;\; product = 27000 = 30^3$$
$$n=5 \;\; 971^3 = 915498611 \;\; (6644263163)_{b8} \;\; product = 373248 = 72^3$$

Base = 9

$$n=1\ \ 1^3=1\ \ (1)_{b9}\ \ product=1=1^3$$
$$n=2\ \ 2^3=8\ \ (8)_{b9}\ \ product=8=2^3$$
$$n=3\ \ 7^3=343\ \ (421)_{b9}\ \ product=8=2^3$$
$$n=4\ \ 16^3=4096\ \ (5551)_{b9}\ \ product=125=5^3$$
$$n=5\ \ 178^3=5639752\ \ (11545251)_{b9}\ \ product=1000=10^3$$

At a glance, there are no more evident surprizes.

C. Asymptotic density of solutions

Need to do more scientific estimate?
My very rough estimate of the density is $(3*2^{log_{10}(n)/3}/n)*(9/10)^{log_{10}(n^3)}$ First multiplier: average value of product of digits of n^3 is on the order of $n^3/2^{log_{10}(n)}$; Second multiplier: zeros with probability 1/10 at every digit.

If my estimate of the density of integers whose product of digits of its cube is also a cube is at least qualitatively correct, then the number of such integers may be finite in some bases: The coefficient $(9/10)^{log_{10}(n^3)}=1/(10/9)^{3log_{10}(n)}=1/n^{3log_{10}(10/9)}$ The exponent $3log_{10}(10/9)=0.13727<1$, so $\sum_{n=1}^{\infty}1/n^{3log_{10}(10/9)}$ diverges. But in base 3, the exponent $3log_3(3/2)=1.1073>1$, so $\sum_{n=1}^{\infty}1/n^{3log_3(3/2)}$ converges; then the total number of such integers is finite with the probability 1. But it only works if such integers occur randomly, or quasi-randomly, which needs to be proven. In the base 3 particularly, the only such integer found clearly isn't quite random.

D. 4th and 5th powers

We also tried 4th and 5th powers - that is integers whose product of digits of its 4th or 5th power is also respectively 4th or 5th power.
Some examples of 4th power.
Base = 3

$$n=1\ \ 1^4=1\ \ (1)_{b3}\ \ product=1=1^4$$
$$n=2\ \ 374^4=19565295376\ \ (1212111112112222121111)_{b3}\ \ product=256=4^4$$

Similar to the 3rd power. Need to prove it, or give some good reasoning
Base = 4

$$n=1\ \ 1^4=1\ \ (1)_{b4}\ \ product=1=1^4$$

It's easy to see why there are no more integers whose product of digits of its 4th power is also 4th power - for any n, $mod(n^4,4^2)=0$ or 1, so the second digit from the right of n^4 is always 0 in base 4.

Base = 5

$$n=1\ \ 1^4=1\ \ (1)_{b5}\ \ product=1=1^4$$
$$n=2\ \ 23^4=279841\ \ (32423331)_{b5}\ \ product=1296=6^4$$
$$n=3\ \ 208^4=1871773696\ \ (12313133224241)_{b5}\ \ product=20736=12^4$$

Seems on the right - except 1, all multiples of 6 - need to prove; the same at bases 7 and 8
Base = 10

$$n=1\ \ 1^4=1\ \ product=1=1^4$$
$$n=2\ \ 118^4=193877776\ \ product=3111696=42^4$$
$$n=3\ \ 144^4=429981696\ \ product=1679616=36^4$$
$$n=4\ \ 211^4=1982119441\ \ product=20736=12^4$$
$$n=5\ \ 427^4=33243864241\ \ product=331776=24^4$$

Seems on the right - except 1, all multiples of 12 - need to prove
Some examples of 5th power.
Base = 3

$$n = 1 \;\; 1^5 = 1 \;\; (1)_{b3} \;\; product = 1 = 1^5$$

Need proof or explanation - why no more in base 3?

Base = 4

$$n = 1 \;\; 1^5 = 1 \;\; (1)_{b4} \;\; product = 1 = 1^5$$

Need proof or explanation - why no more in base 4?

The reason that, except 1, there are no 5th powers, product of digits of which is also 5th power, in bases 3 and 4, may be in asymptotic density of such numbers (discussed above for the 3rd power), which is lower for smaller bases, and it sums to a finite number of such numbers

Base = 5

$n = 1 \;\; 1^5 = 1 \;\; (1)_{b5} \;\; product = 1 = 1^5$
$n = 2 \;\; 33302^5 = 40959211016381193864032 \;\; (133442211231122314342141112122112)_{b5} \;\; product = 254803968 = 48^5$
$n = 3 \;\; 266498^5 = 1344218197841297128241319968 \;\; (33214133312222111241134441221142342143333)_{b5}$
$product = 1981355655168 = 288^5$

Base = 10

$$n = 1 \;\; 1^5 = 1 \;\; product = 1 = 1^5$$
$$n = 2 \;\; 382^5 = 8134236862432 \;\; product = 7962624 = 24^5$$
$$n = 3 \;\; 394^5 = 9494696984224 \;\; product = 1934917632 = 72^5$$

III. CONCLUSIONS

In this work we studied cubes of integers in base 10, such that the product of digits of this cube is itself a cube.
Need to expand
Thanks who helped...
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