GAUSSIAN INTEGER SOLUTIONS FOR THE FIFTH POWER TAXICAB NUMBER PROBLEM

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Abstract. The famous open problem of finding positive integer solutions to $a^5 + b^5 = c^5 + d^5$ is considered, and related solutions are found in two distinct settings: firstly, where a and b are both positive integers with c and d both Gaussian integers; secondly, where all of a, b, c, and d are Gaussian integers.

1. Introduction

A famous open question [3] is the solution in positive integers of

$$w^5 + x^5 = y^5 + z^5. \tag{1.1}$$

It however has solutions in Gaussian integers. A few examples:

$$(1 + 2i)^5 + (4 - 7i)^5 = (-3 - 6i)^5 + (8 + i)^5 \tag{1.2}$$

$$(3 + 10i)^5 + (9 - 6i)^5 = (6 + 5i)^5 + (6 - i)^5 \tag{1.3}$$

$$(4 + 45i)^5 + (36 - 5i)^5 = (45 + 4i)^5 + (-5 + 36i)^5 \tag{1.4}$$

While not resolving this for integers generally, we give an infinite set of solutions with integers w and x where the right side y and z are Gaussian integers. Also, we give an infinite set of solutions where all of w, x, y, and z are Gaussian integers. The solutions to

$$w^n + x^n = y^n + z^n, \tag{1.5}$$

for $n = 4$ are well known and date back to Euler [5], and the case where $n = 3$ is solved by the well known and celebrated "Taxicab numbers" named after the famous Hardy and Ramanujan anecdote. (See Hardy [2].).

2. Solutions where a and b are both positive integers with c and d both Gaussian integers

Our first result is encapsulated in the

Theorem 2.1. *If the Pell number sequence is 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378,...; then an infinite sequence of solutions to* (1.1) *is given by:*

$$3^5 + 1^5 = (2 + i3)^5 + (2 - i3)^5, \tag{2.1}$$

$$13^5 + 11^5 = (12 + i17)^5 + (12 - i17)^5, \tag{2.2}$$

$$71^5 + 69^5 = (70 + i99)^5 + (70 - i99)^5, \tag{2.3}$$

$$409^5 + 407^5 = (408 + i577)^5 + (408 - i577)^5, \tag{2.4}$$

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(2.5) $$2379^5 + 2377^5 = (2378 + i3363)^5 + (2378 - i3363)^5,$$

and so on where for P_n the nth Pell number, the nth equation is

(2.6)
$$(P_{2n+3}+1)^5+(P_{2n+3}-1)^5 = (P_{2n+3}+i(P_{2n+3}+P_{2n+2}))^5+(P_{2n+3}-i(P_{2n+3}+P_{2n+2}))^5.$$

It does seem interesting that the ancient Pell number sequence should figure so neatly in the above set of solutions, with integers on the left, Gaussian integers on the right. The proof of Theorem 2.1 is by simple expansion.

3. Solutions where all of a, b, c and d are Gaussian integers

Our second result requires the following identity,

Lemma 3.1. *For all real values of a, b, c,*

(3.1) $$(a+b+ic)^5+(a-b-ic)^5-(a-b+ic)^5-(a+b-ic)^5 = 80abc(a^2+b^2-c^2)i.$$

We see that every Pythagorean triple a, b, c yields a zero on the right side of the lemma, and hence a Gaussian integer solution of $A^5 + B^5 = C^5 + D^5$. This proves the

Theorem 3.1. *Every Pythagorean triple a, b, c implies a Gaussian integer solution of $A^5 + B^5 = C^5 + D^5$.*

Some examples with primitive triples are:
$(4, 3, 5)$ leads to $(7 + 5i)^5 + (1 - 5i)^5 = (7 - 5i)^5 + (1 + 5i)^5$,
$(12, 5, 13)$ leads to $(17 + 13i)^5 + (7 - 13i)^5 = (7 + 13i)^5 + (17 - 13i)^5$,
$(15, 8, 17)$ leads to $(23 + 17i)^5 + (7 - 17i)^5 = (7 + 17i)^5 + (23 - 17i)^5$.

Theorem 3.2. *The equation*

(3.2) $$(a + bi)^5 + (c + di)^5 = (b + ai)^5 + (d + ci)^5$$

has an infinite number of solutions.

PROOF. The equation (3.2) is equivalent to

(3.3) $$(a - b)^5 - 20a^2b^2(a - b) + (c - d)^5 - 20c^2d^2(c - d) = 0,$$

which is satisfied if $a-b+c-d = 0$ and $ab+cd = 0$. Let a be arbitrary. Substituting $b = a+c-d$ from the first equation into the second, we get $d = a+2ac/(a-c)$, which is integer at least for $c = 0, a \pm 1, a \pm 2, 2a, a \pm 2a$. So for every integer a equation (3.2) has a few solutions, altogether amounting to an infinite number. END OF PROOF.

A very similar

Theorem 3.3. *The equation*

(3.4) $$(a + bi)^5 + (b + ai)^5 = (c + di)^5 + (d + ci)^5$$

has an infinite number of solutions.

Examples of solutions in the shape of (3.2) and (3.4):

(3.5) $$(3 + 28i)^5 + (4 - 21i)^5 = (28 + 3i)^5 + (-21 + 4i)^5$$

(3.6) $$(2 + 3i)^5 + (3 + 2i)^5 = (6 - i)^5 + (-1 + 6i)^5$$

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