UCDAVIS

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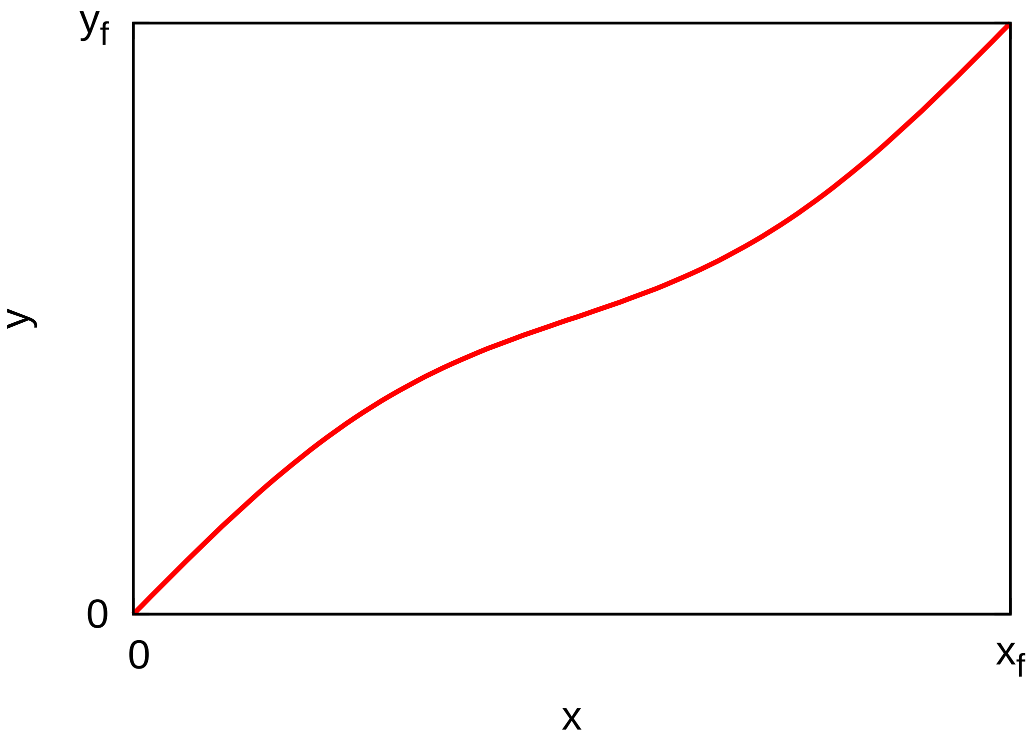
# Integration by Parts

# Outline

- Definite Integral by Parts
- Indefinite Integral by Parts
- General Formula
- Examples
- History
- QUIZ!

# Definite Integral by Parts

- **Problem : Integrate function $\mathbf{y = y(x)}$**
- 



# Definite Integral by Parts

- **Red Area** $= \int_{x=0}^{x_f} y \, dx$
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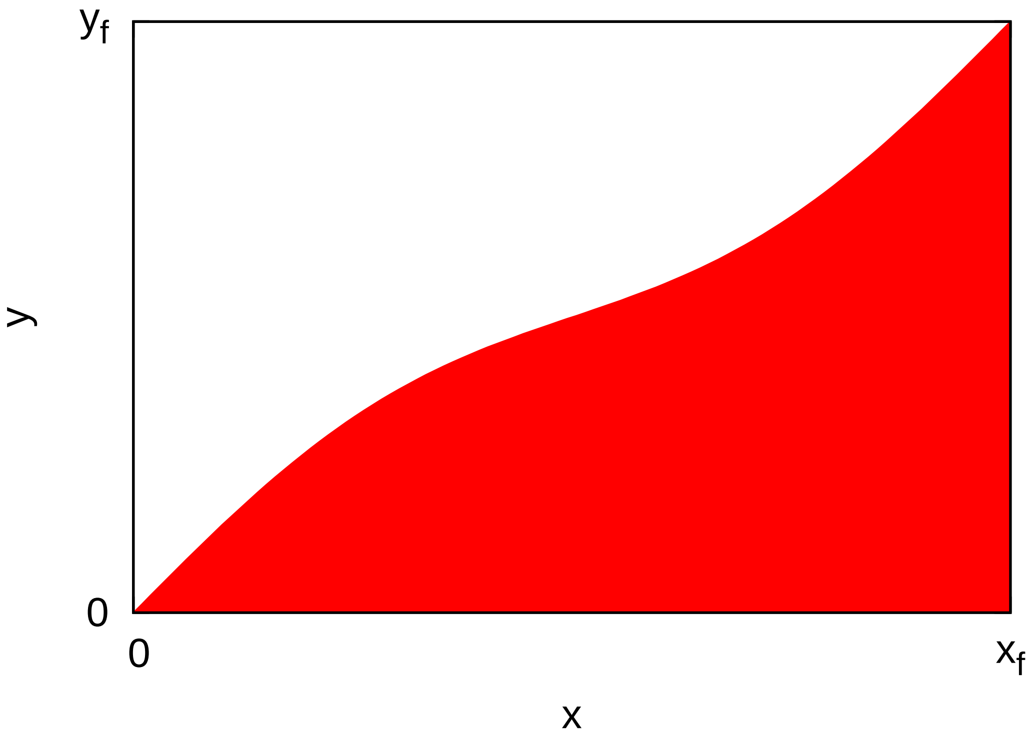

# Definite Integral by Parts

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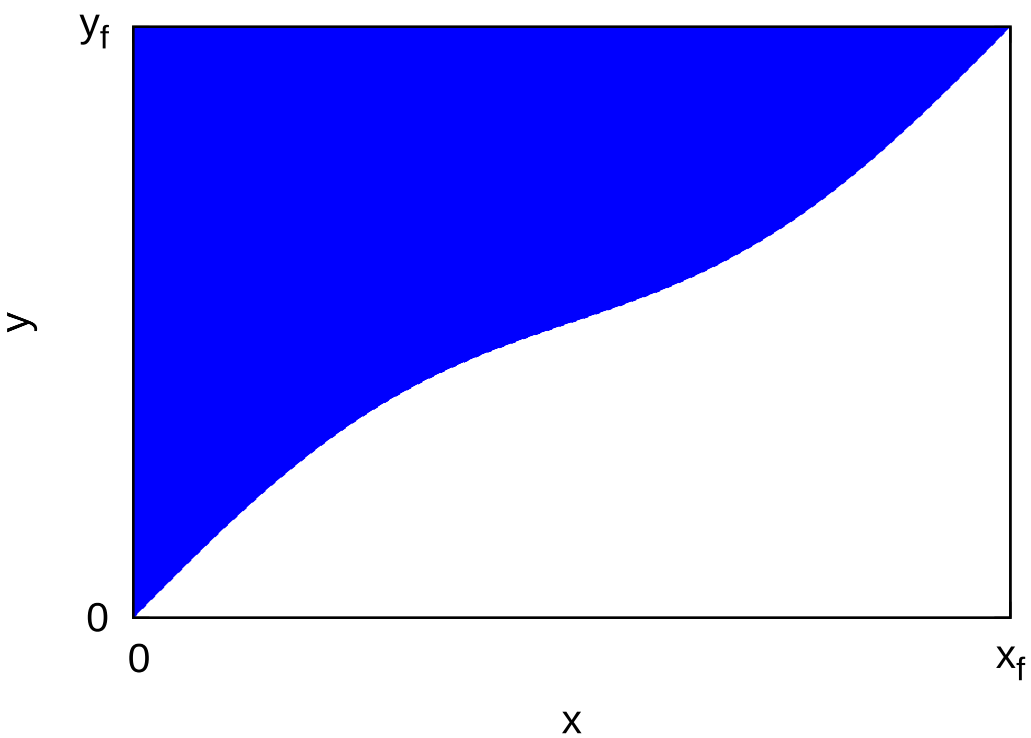
• **Blue Area** $= \int_{y=0}^{y_f} x \, dy$



# Definite Integral by Parts

- **Red Area** $= \int_{x=0}^{x_f} y \, dx$
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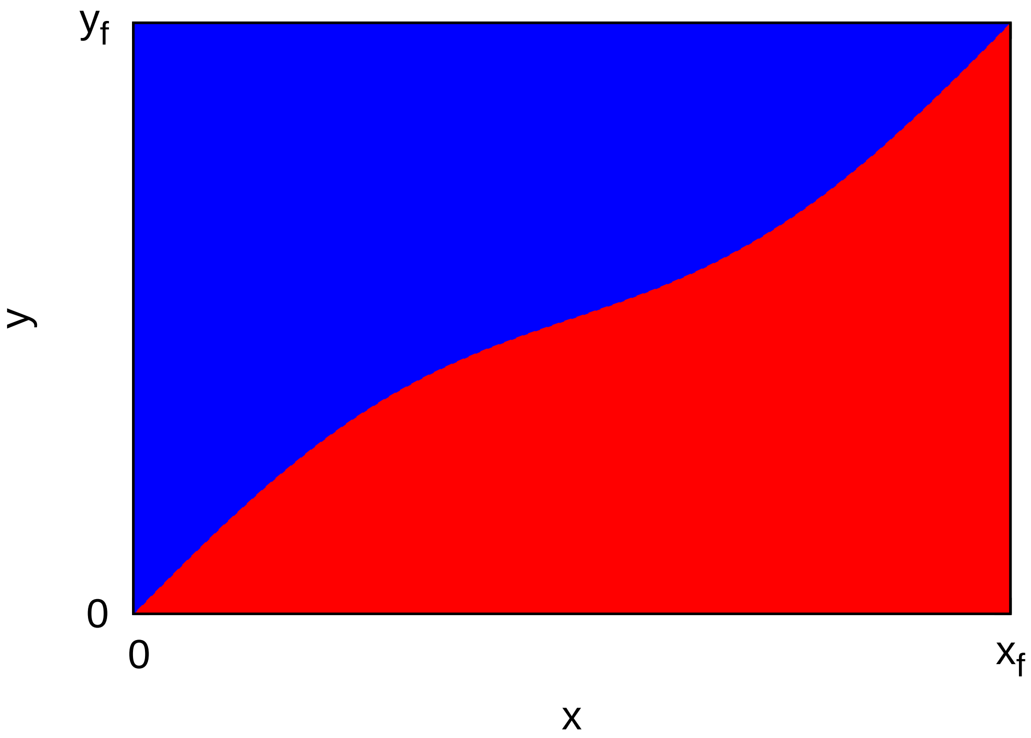
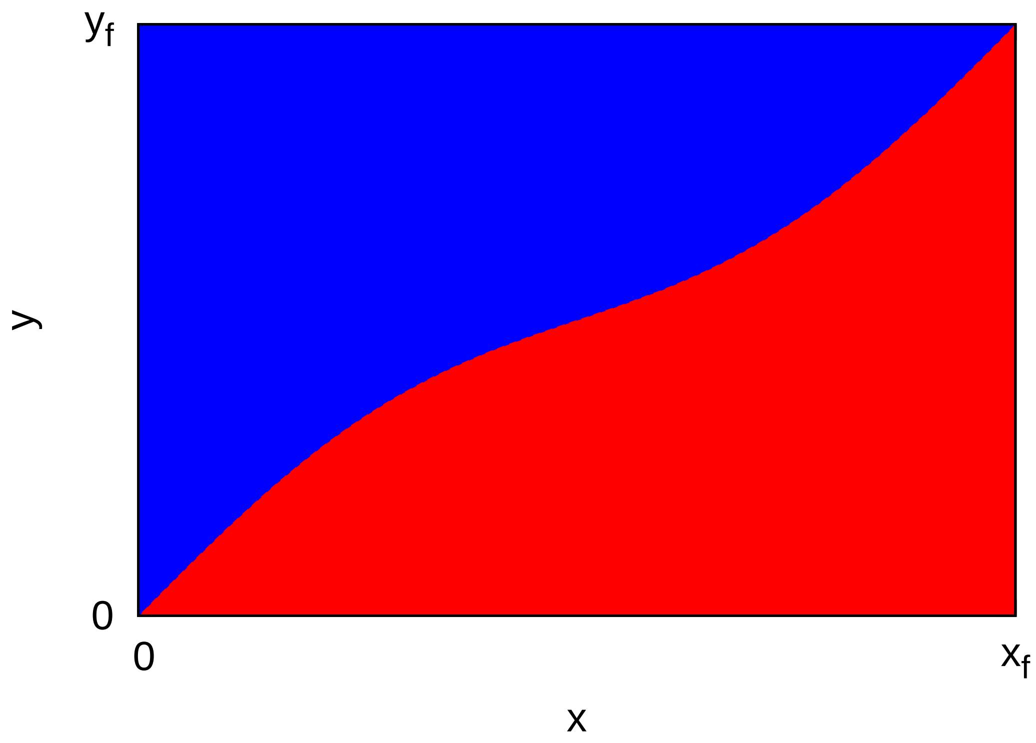


# Definite Integral by Parts

- $$\textbf{Red Area} = \int_{x=0}^{x_f} y\,dx$$
- $$\textbf{Blue Area} = \int_{y=0}^{y_f} x\,dy$$
- $$\textbf{Red Area} + \textbf{Blue Area} = \int_{x=0}^{x_f} y\,dx + \int_{y=0}^{y_f} x\,dy = \ ?$$

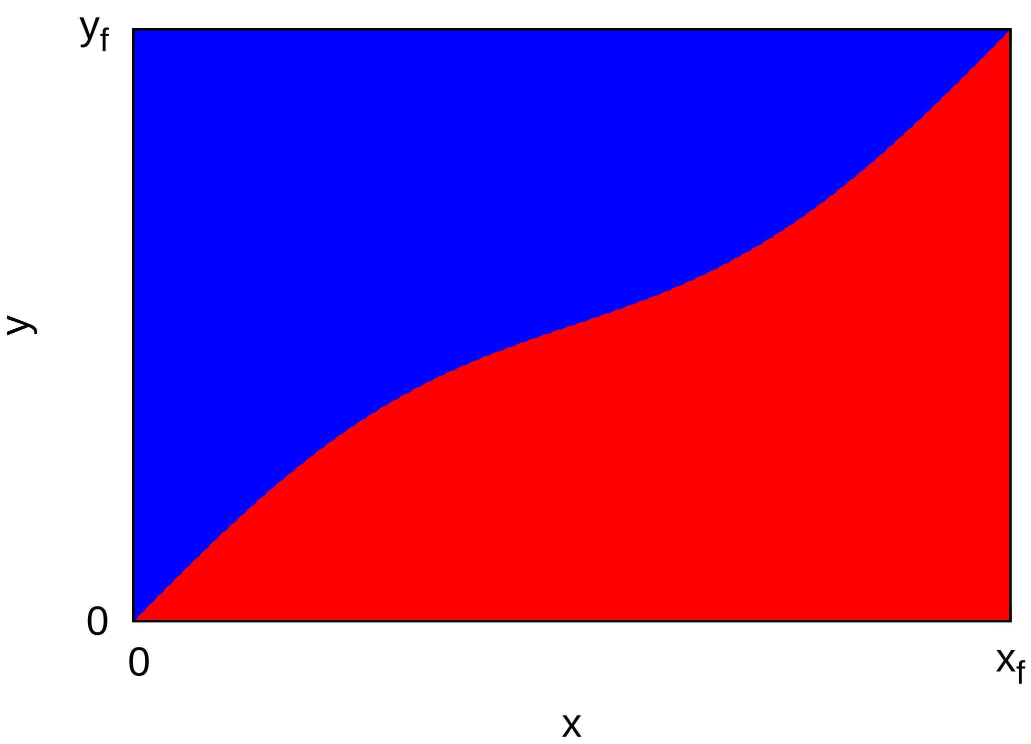


# Definite Integral by Parts

$$\bullet\textbf{Red Area} = \int_{x=0}^{x_f} y\, dx$$

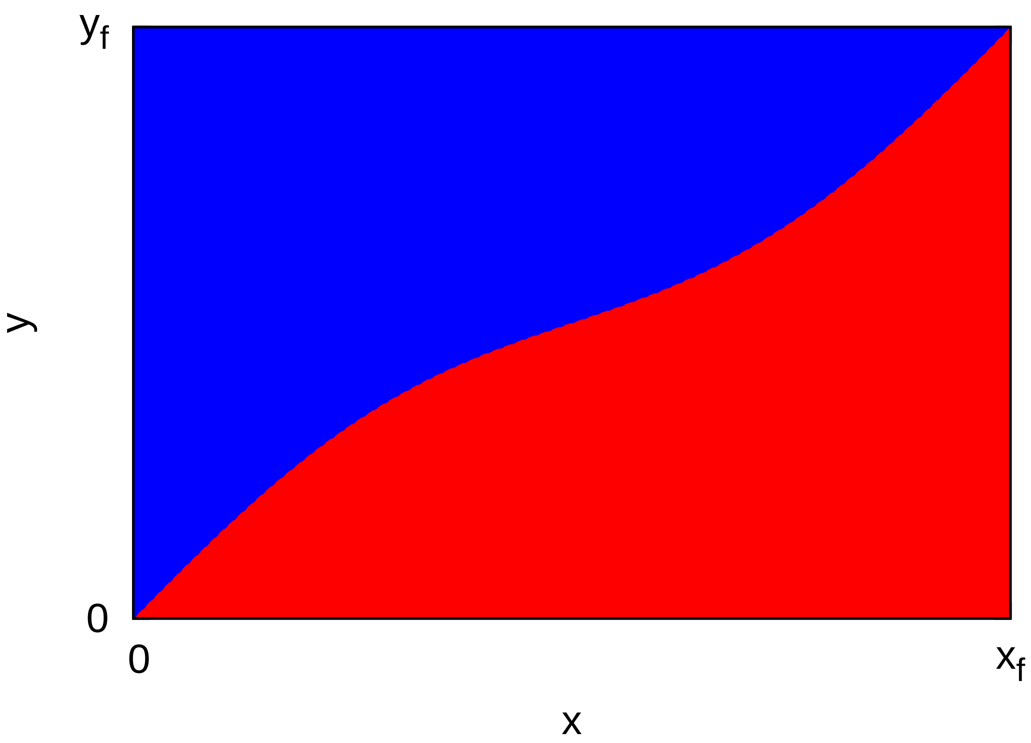
$$\bullet\textbf{Blue Area} = \int_{y=0}^{y_f} x\, dy$$

$$\bullet\textbf{Red Area} + \textbf{Blue Area} = \int_{x=0}^{x_f} y\, dx + \int_{y=0}^{y_f} x\, dy = x_f \cdot y_f$$



# Definite Integral by Parts

- $\textbf{Red Area} + \textbf{Blue Area} = \int_{x=0}^{x_f} y\,dx + \int_{y=0}^{y_f} x\,dy = x_f \cdot y_f$
- $\int_{x=0}^{x_f} y\,dx = x_f \cdot y_f - \int_{y=0}^{y_f} x\,dy$



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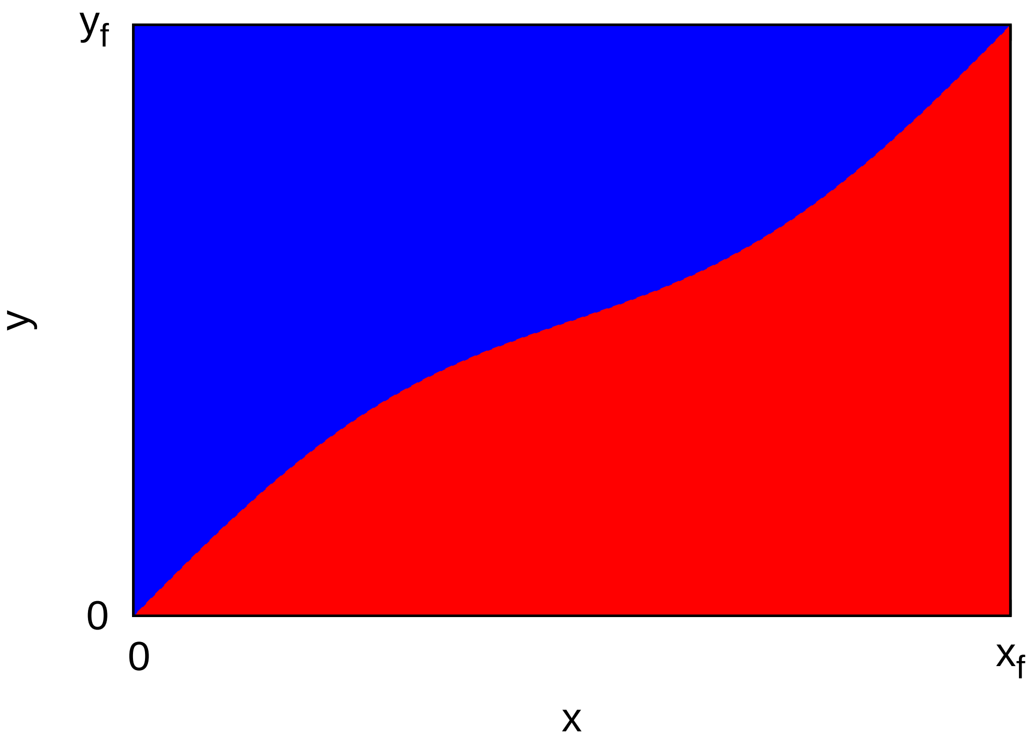


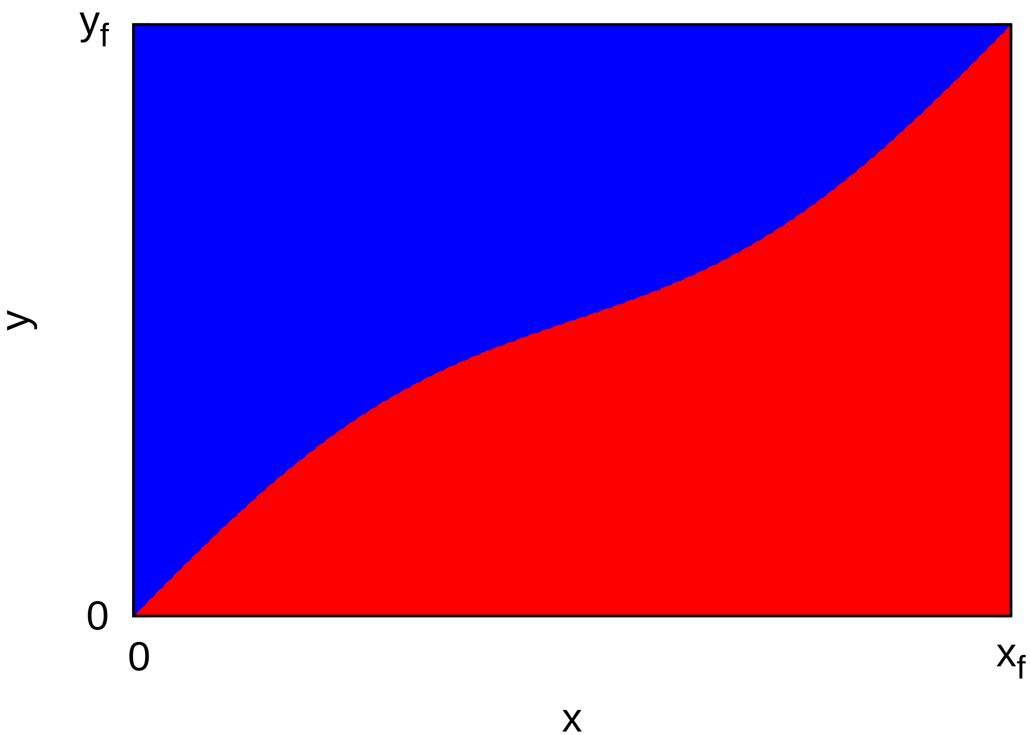
# Indefinite Integral by Parts

## Definite Integral

$$\int_{x=0}^{x_f} y\,dx = x_f \cdot y_f - \int_{y=0}^{y_f} x\,dy$$

## Indefinite Integral

$$\int y\,dx = x \cdot y - \int x\,dy$$



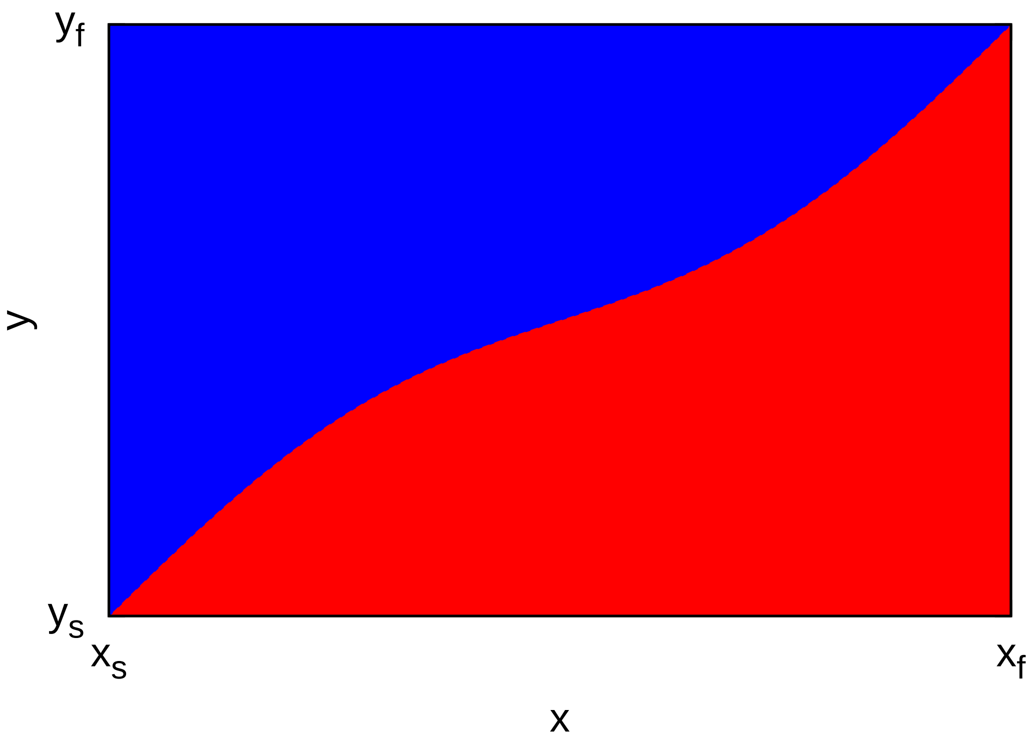
# Definite Integral by Parts

## Indefinite Integral

$$\int \mathbf{y}\, \mathbf{dx} = \mathbf{x} \cdot \mathbf{y} - \int \mathbf{x}\, \mathbf{dy}$$

## Definite Integral

$$\int_{\mathbf{x}=\mathbf{x_s}}^{\mathbf{x_f}} \mathbf{y}\, \mathbf{dx} = \mathbf{x_f} \cdot \mathbf{y_f} - \mathbf{x_s} \cdot \mathbf{y_s} - \int_{\mathbf{y}=\mathbf{y_s}}^{\mathbf{y_f}} \mathbf{x}\, \mathbf{dy}$$



# General Formula

$$\int \mathbf{y}\ \mathbf{dx} = \ \mathbf{x} \cdot \mathbf{y} - \int \mathbf{x}\ \mathbf{dy}$$

$x = u(t)$
$y = v(t)$

$$\int \mathbf{y}\ \mathbf{dx} = \ \mathbf{x} \cdot \mathbf{y} - \int \mathbf{x}\ \mathbf{dy}$$

$$\int \mathbf{u}(\mathbf{t})\ \mathbf{d}(\mathbf{v}(\mathbf{t})) = \ \mathbf{u}(\mathbf{t}) \cdot \mathbf{v}(\mathbf{t}) - \int \mathbf{v}(\mathbf{t})\ \mathbf{d}(\mathbf{u}(\mathbf{t}))$$

$$\int \mathbf{u}(\mathbf{t})\ \mathbf{v}'(\mathbf{t})\ \mathbf{dt} = \ \mathbf{u}(\mathbf{t}) \cdot \mathbf{v}(\mathbf{t}) - \int \mathbf{v}(\mathbf{t})\ \mathbf{u}'(\mathbf{t})\ \mathbf{dt}$$

# General Formula

Need to calculate

$$\int \mathbf{f(t)\ dt}$$

Recognize that $f(t)$ may be represented

$$f(t) \;=\; u(t)\, v'(t)$$

for some $u(t)$ and $v(t)$. Then

$$\int \mathbf{f(t)\ dt} = \int \mathbf{u(t)\ v'(t)\ dt} = \mathbf{u(t) \cdot v(t)} - \int \mathbf{v(t)\ u'(t)\ dt}$$

Possibly the integral

$$\int \mathbf{v(t)\ u'(t)\ dt}$$

is easier than $\int \mathbf{f(t)\ dt}$

# Example

$$\int \cos(\mathbf{t})\ \mathbf{t}\ \mathbf{dt}$$

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Try:

$u(t) = cos(t)$

$v'(t) = t \Rightarrow v(t) = \frac{t^2}{2}$

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Not good - integral more complicated

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Try:
$u(t) = t$
$v'(t) = \cos(t) \Rightarrow v(t) = \sin(t)$

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Try:
$u(t) = t$
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$$\int \cos(\mathbf{t})\ \mathbf{t}\ \mathbf{dt} = \mathbf{t}\ \sin(\mathbf{t}) - \int \sin(\mathbf{t})\ \mathbf{dt} = \mathbf{t}\ \sin(\mathbf{t}) + \cos(\mathbf{t}) + C$$

# Example

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Success!

# History - Who invented integration by parts?

An early use of integration by parts appears in the work of Brook Taylor. Methodus Incrementorum (London, 1715), Prop XI, Theor IV, one of his investigations of series.

Brook Taylor is known for Taylor series
https://en.wikipedia.org/wiki/Brook_Taylor

# Next: Numerical Integration using Monte Carlo method

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This File

http://zujev.physics.ucdavis.edu/Lessons/IP.pdf

# QUIZ: Calculate using Integration by Parts

(1) $$\int t \ \sin(t)\ dt$$

(2) $$\int x\ e^x\ dx$$

(3) $$\int \ln(x)\ dx$$

(4) $$\int_{x=0}^{\infty} x\ e^{-x}\ dx$$

(5) $$\int x^2 e^x\ dx$$

(6) $$\int \cos(x)\ e^x\ dx$$

# QUIZ: Calculate using Integration by Parts

$$(1) \qquad \int t \ \sin(t)\ dt \ = \ -t\cos(t) + \sin(t) + C$$

$$(2) \qquad \int x\ e^x\ dx \ = \ (x-1)e^x + C$$

$$(3) \qquad \int \ln(x)\ dx \ = \ x(\ln(x) - 1) + C$$

$$(4) \qquad \int_{x=0}^{\infty} x\ e^{-x}\ dx \ = \ 1$$

$$(5) \qquad \int x^2 e^x \ = \ (x^2 - 2x + 2)e^x$$

$$(6) \qquad \int \cos(x)\ e^x \ = \ \frac{1}{2}\left(\sin(x) + \cos(x)\right) e^x$$